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A new class of realisations of the Lie algebra sp(n, R)

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Abstract. The method for constructing boson realisations of semisimple Lie algebras formulated in our previous paper is applied to the case of sp(n, R). The realisations obtained are expressed by means of certain recurrence formulae in terms of $r(2n-\frac{3}{2}r+\frac{1}{2})$ canonical pairs and generators of the subalgebra gl(r, R)$\oplus$sp($n-r$, R), where r is a fixed number equal to $1, 2, \ldots, n$. Each of the realisations is skew-Hermitian with respect to the natural involution and Schur irreducible.

1. Introduction

The real symplectic algebra sp(n, R), which is the algebra of the group of linear canonical transformations in $2n$-dimensional phase space, plays an outstanding role in many physical problems (Moshinsky and Quesne 1971, Wybourne 1974). This is why various types of representations of this algebra are interesting. In this paper, we are going to concentrate our attention on a purely algebraic method of constructing the representations. The central notion is that of the canonical (or boson) realisation which means an expression of elements of sp(n, R) by means of polynomials in quantum canonical variables p_i, q_i, such that the commutation relations are preserved. For the physical relevance of such canonical realisations of sp(n, R) see Deenen and Quesne (1984a) and other references quoted therein.

Explicit forms of realisations for sp(n, R) have been constructed by Deenen and Quesne (1984b) and Rowe (1984) using the method of coherent state representation (Dobaczewski 1981). These realisations are defined by means of $\frac{1}{2}n(n+1)$ canonical pairs and of generators of a subalgebra gl(n, R). Another class of realisations has been described by Havlíček and Lassner (1976); in their paper, realisations of sp(n, R) in terms of $(2n-1)$ canonical pairs and of generators of sp($n-1$, R) were obtained.

We have formulated recently a method (Burdík 1985), which enables us to construct wide families of canonical realisations of a semisimple Lie algebra g starting from induced representations of g with respect to a suitable subalgebra. In the present paper, we apply this method to the case of sp(n, R). For any $r = 1, 2, \ldots, n$, we construct recurrence formulae which give realisations of sp(n, R) in terms of $r(2n-\frac{3}{2}r+\frac{1}{2})$ canonical pairs and of generators of the subalgebra gl(r, R)$\oplus$sp($n-r$, R). Both the above-mentioned types of realisations appear to be particular cases for $r = n$ and $r = 1$, respectively.

2. Preliminaries

The algebra $g \equiv sp(n, R)$ is the $n(2n+1)$-dimensional real Lie algebra. We choose a basis consisting of $n(2n+1)$ generators $X_{ij} = -\varepsilon_i\varepsilon_j X_{-j,-i}$, $i, j = -n, \ldots, -1, 1, \ldots, n$ satisfying the commutation rules

$$[X_{ij}, X_{kl}] = \delta_{jk}X_{il} - \delta_{il}X_{kj} + \varepsilon_i\varepsilon_j\delta_{j,-l}X_{k,-i} - \varepsilon_i\varepsilon_j\delta_{i,-k}X_{-j,l} \tag{1}$$

where $\varepsilon_i \equiv \operatorname{sgn} i$.

For any $r = 1, 2, \ldots, n-1$, we define

$$b_r = \sum_{i=1}^{r} X_{ii}$$

and any such b_r gives the following decomposition of the algebra $sp(n, R)$:

$$\begin{aligned} g &= n_+^{b_r} \oplus g_0^{b_r} \oplus n_-^{b_r} \\ n_+^{b_r} &\equiv R\{X \in g, [b_r, X] = \alpha_X^r X, \text{ where } \alpha_X^r > 0\} \\ g_0^{b_r} &\equiv \{X \in g, [b_r, X] = 0\} \\ n_-^{b_r} &\equiv R\{X \in g, [b_r, X] = -\alpha_X^r X, \text{ where } \alpha_X^r > 0\}. \end{aligned} \tag{2}$$

This decomposition will be used as a starting point for our construction. More details about properties of such decompositions can be found in Burdík (1985).

The Weyl algebra W_{2N_r} is the complex associative algebra with identity generated by $2N_r$ elements

$$\begin{aligned} &p_{it}, p_{i,-t}, p_{i,-j} \qquad && p_{i,-j} = \varepsilon_i\varepsilon_j p_{j,-i} \\ &q_{it}, q_{i,-t}, q_{i,-j} \qquad && q_{i,-j} = \varepsilon_i\varepsilon_j q_{j,-i} \end{aligned} \tag{3}$$

where $i, j = 1, 2, \ldots, r$, $t = r+1, r+2, \ldots, n$. They satisfy the commutation relations

$$\begin{aligned} &[p_{it}, q_{js}] = \delta_{ij}\delta_{st} \\ &[p_{i,-t}, q_{j,-s}] = \delta_{ij}\delta_{st} \\ &[p_{i,-j}, q_{k,-l}] = \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{(1+\delta_{ij})}. \end{aligned} \tag{4}$$

Let g_0 be a real Lie algebra. By $\tilde{g}_0$ we denote its complexification; furthermore, $U(\tilde{g}_0)$ is the enveloping algebra of this complexification.

Definition. A realisation of g is a homomorphism

$$\tau : g \to W_{2N_r} \otimes U(\tilde{g}_0). \tag{5}$$

The homomorphism τ extends naturally to the homomorphic mapping (denoted by the same symbol τ) of the enveloping algebra $U(\tilde{g})$ into $W_{2N_r} \otimes U(\tilde{g}_0)$.

Definition. Let $Z(\tilde{g})$ be the centre of $U(\tilde{g})$. A realisation is called Schurean (Schur irreducible or simply Schur realisation) if all the central elements $C \in Z(\tilde{g})$ are realised by $1 \otimes C_0$ where C_0 are central elements of the enveloping algebra $U(\tilde{g}_0)$.

In view of possible applications to the representation theory, we introduce the involution '+' in W_{2N_r} be means of the following relations:

$$q^+_{\alpha\beta} = -q_{\alpha\beta} \qquad p^+_{\alpha\beta} = p_{\alpha\beta} \tag{6a}$$

where α, β run over all the allowed values of the indices. Similarly, the involution '+' on $U(\tilde{g}_0)$ is defined by

$$Y^+ = -Y \qquad \text{for } Y \in g_0. \tag{6b}$$

These involutions define naturally an involution on $W_{2N_r} \otimes U(\tilde{g}_0)$:

$$\left(\sum_j \alpha_j \pi_j \otimes g_j\right)^+ = \sum_j \bar{\alpha}_j \pi_j^+ \otimes g_j^+ \tag{6c}$$

where $\pi_j \in W_{2N_r}$ and $g_j \in U(\tilde{g}_0)$.

Definition. Let g be a real Lie algebra and let '+' be the involution on $W_{2N_r} \otimes U(\tilde{g}_0)$ described above. A realisation τ of g on $W_{2N_r} \otimes U(\tilde{g}_0)$ is called skew-Hermitian if, for all elements $X \in g$, the following relations hold:

$$(\tau(X))^+ = -\tau(X). \tag{7}$$

3. Construction of realisations

Using the commutation relations (1) we can bring the decomposition (2) into the form

$$\begin{aligned} n_+^{b_r} &= R\{X_{it}, X_{i,-t}, X_{i,-j}\} \\ g_0^{b_r} &= R\{X_{ij}, X_{st}, X_{s,-t}, X_{-t,s}\} \sim \mathrm{gl}(r, R) \oplus \mathrm{sp}(n-r, R) \\ n_-^{b_r} &= R\{X_{ti}, X_{-t,i}, X_{-j,i}\} \end{aligned} \tag{8}$$

where again $i, j = 1, 2, \ldots, r$ and $s, t = r+1, r+2, \ldots, n$. Evidently, the set $\{X_{it}, X_{i,-t}, X_{i,-j}\colon i \leqslant j,\ i, j = 1, 2, \ldots, r;\ t = r+1, \ldots, n\}$ is a basis in $n_+^{b_r}$. We write the elements of this basis as the matrix

$$\begin{vmatrix} X_{1,r+1} & X_{1,r+2} & \cdots & X_{1,n} \\ \vdots & & & \\ X_{r,r+1} & X_{r,r+2} & \cdots & X_{r,n} \\ X_{1,-1} & X_{1,-2} & \cdots & X_{1,-n} \\ & X_{r,-r} & \cdots & X_{r,-n} \end{vmatrix} \tag{9}$$

and assume that the basis is ordered lexicographically. The monomials of $U(\tilde{n}_+^{b_r})$ can then be written as the matrices

$$\begin{vmatrix} n_{1,r+1} & \cdots & n_{1,n} \\ \vdots & & \\ n_{r,r+1} & \cdots & n_{r,n} \\ n_{1,-1} & \cdots & n_{1,-n} \\ n_{r,-r} & \cdots & n_{r,-n} \end{vmatrix} \equiv (X_{1,r+1}^{n_{1,r+1}} \cdots X_{1,n}^{n_{1,n}}) \cdots (X_{r,r+1}^{n_{r,r+1}} \cdots X_{r,n}^{n_{r,n}}) \times (X_{1,-1}^{n_{1,-1}} \cdots X_{1,-n}^{n_{1,-n}}) \cdots (X_{r,-r}^{n_{r,-r}} \cdots X_{r,-n}^{n_{r,-n}}) \tag{10}$$

where, of course, n_{it}, $n_{i,-t}$, $n_{i,-j}$ belongs to N_0, the set of all non-negative integers, for any $i, j = 1, 2, \ldots, r$ and $t = r+1, r+2, \ldots, n$.

Now we are able to apply the general construction described in Burdík (1985). Let σ_r be an auxiliary representation of the algebra $g_0^r \oplus n_-^{b_r}$ on a vector space V such that

$$\begin{aligned} &\sigma_r(n_-^{b_r}) = 0 \\ &\sigma_r|g_0^{b_r} \quad \text{is faithful.} \end{aligned} \tag{11}$$

We denote by W the carrier space of the induced representation $\rho_r = \mathrm{ind}(g, \sigma_r)$. If $v_1, \ldots, v_d$ is a basis in the space V, then the vectors

$$\left|\begin{matrix} n_{1,r+1} & \cdots & n_{1,n} \\ \vdots & & \\ n_{r,r+1} & \cdots & r_{r,n} \\ n_{1,-1} & \cdots & n_{1,-n} \\ n_{r,-r} & \cdots & n_{r,-n} \end{matrix}\right| \otimes v_i \tag{12}$$

form a basis in W.

We define the creation and annihilation operators $\bar{a}_{it}$, a_{js} on the space W in the following way:

$$\begin{aligned} &\bar{a}_{it}\left|\begin{matrix} n_{1,r+1} & \cdots & n_{1,n} \\ \cdots & n_{it} & \cdots \\ n_{r,r+1} & \cdots & n_{r,n} \\ n_{1,-1} & \cdots & n_{1,-n} \\ n_{r,-r} & \cdots & n_{r,-n} \end{matrix}\right| \otimes v_i \equiv \left|\begin{matrix} n_{1,r+1} & \cdots & n_{1,n} \\ \cdots & n_{it}+1 & \cdots \\ n_{r,r+1} & \cdots & r_{r,n} \\ n_{1,-1} & \cdots & n_{1,-n} \\ n_{r,-r} & \cdots & n_{r,-n} \end{matrix}\right| \otimes v_i \\ &a_{js}\left|\begin{matrix} n_{1,r+1} & \cdots & n_{1,n} \\ \cdots & n_{js} & \cdots \\ n_{r,r+1} & \cdots & n_{r,n} \\ n_{1,-1} & \cdots & n_{1,-n} \\ n_{r,-r} & \cdots & n_{r,-n} \end{matrix}\right| \otimes v_i \equiv n_{js}\left|\begin{matrix} n_{1,r+1} & \cdots & n_{1,n} \\ \cdots & n_{js}-1 & \cdots \\ n_{r,r+1} & \cdots & n_{r,n} \\ n_{1,-1} & \cdots & n_{1,-n} \\ n_{r,-r} & \cdots & n_{r,-n} \end{matrix}\right| \otimes v_i \end{aligned} \tag{13a}$$

and similarly we define the operators $\bar{a}_{i,-t}$, $a_{i,-t}$, $\bar{a}_{i,-j}$, $i \leq j$. Furthermore, we define the operators $\tilde{X}_{ik}$, $i, k = 1, 2, \ldots, r$; $\tilde{X}_{st}$, $\tilde{X}_{s,-t}$, $\tilde{X}_{-s,t}$, $s, t = r+1, r+2, \ldots, n$, by the relations

$$\begin{aligned} &\tilde{X}_{ik} = 1 \otimes \sigma_r(X_{ik}) \qquad &&\tilde{X}_{st} = 1 \otimes \sigma_r(X_{st}) \\ &\tilde{X}_{s,-t} = 1 \otimes \sigma_r(X_{s,-t}) \qquad &&\tilde{X}_{-s,t} = 1 \otimes \sigma_r(X_{-s,t}). \end{aligned} \tag{13b}$$

According to theorem 1 of Burdík (1985) the induced representation $\rho_r = \mathrm{ind}(g, \sigma_r)$ can be rewritten using the above defined operators (13a) and 13b). We get the formulae

$$\rho_r(X_{ij}) = \sum_{s=r+1}^{n} (\bar{a}_{is}a_{js} + \bar{a}_{i,-s}a_{j,-s}) + \bar{a}_{i,-j}a_{j,-j} + \sum_{k=1}^{r} \bar{a}_{k,-i}a_{k,-j} + \tilde{X}_{ij}$$

$$\rho_r(X_{st}) = \sum_{k=1}^{r} (\bar{a}_{k,-s}a_{k,-t} - \bar{a}_{kt}a_{ks}) + \tilde{X}_{st}$$

$$\rho_r(X_{s,-t}) = \sum_{j=1}^{r}\sum_{k=j}^{r} (\bar{a}_{j,-k}a_{kt} - \bar{a}_{j,-t})a_{js} + \sum_{j=1}^{r}\sum_{k=j+1}^{r} (\bar{a}_{j,-k}a_{ks} - a_{j,-s})a_{jt} + \tilde{X}_{s,-t}$$

$$\rho_r(X_{-s,t}) = \sum_{j=1}^{r}\sum_{k=1}^{j-1}(\bar{a}_{k,-j}a_{k,-t} - \bar{a}_{jt})a_{j,-s} + \sum_{j=1}^{r}\sum_{k=1}^{j}(\bar{a}_{k,-j}a_{k,-s} - a_{js})a_{j,-t} + \tilde{X}_{-s,t} \tag{14}$$

$$\rho_r(X_{r,r+1}) = \bar{a}_{r,r+1}$$

$$\rho_r(X_{r+1,r}) = \sum_{s=r+1}^{n}(\rho_r^*(X_{r+1,s})a_{rs} + a_{r,-s}\tilde{X}_{r+1,s}) - \sum_{j=1}^{r}\rho_r(X_{jr})a_{j,r+1} + 2\bar{a}_{r,-(r+1)}a_{r,-r}$$

where

$$\rho_r^*(X_{r+1,s}) = \sum_{k=1}^{r}\bar{a}_{k,-(r+1)}a_{k,-s} + \tilde{X}_{r+1,s}$$

and in the case $r = n$

$$\rho_n(X_{i,-j}) = \bar{a}_{i,-j}$$

$$\rho_n(X_{-n,n}) = -\sum_{k=1}^{n-1}\left(2\tilde{X}_{kn} + \bar{a}_{k,-k} + \sum_{j=k+1}^{n-1}2\bar{a}_{k,-j}a_{j,-n}\right)a_{k,-n}$$
$$- \sum_{k=1}^{n}4\bar{a}_{k,-n}a_{k,-n}a_{n,-n} - 4a_{n,-n}\tilde{X}_{nn}.$$

The representation of the remaining generators we obtain using the commutation rules (1).

Now the skew-Hermitian realisations sought are obtained easily by replacing the operators in the above expressions by suitable algebraic objects. For details, see Burdík (1985, propositions 1, 2 and theorem 2). They are given by the formulae

$$\tau_r'(X_{ij}) = \sum_{s=r+1}^{n}(q_{is}p_{js} + q_{i,-s}p_{j,-s}) + q_{i,-j}p_{j,-j} + \sum_{k=1}^{r}q_{k,-i}p_{k,-j} + X_{ij} + (n - \tfrac{1}{2}r + \tfrac{1}{2})\delta_{ij}$$

$$\tau_r'(X_{st}) = \sum_{k=1}^{r}(q_{k,-s}p_{k,-t} - q_{kt}p_{ks}) + X_{st}$$

$$\tau_r'(X_{s,-t}) = \sum_{j=1}^{r}\sum_{k=1}^{r}(q_{j,-k}p_{kt} - q_{j,-t})p_{js} + \sum_{j=1}^{r}\sum_{k=j+1}^{r}(q_{j,-k}p_{ks} - q_{j,-s}p_{jt}) + X_{s,-t}$$

$$\tau_r'(X_{-s,t}) = \sum_{j=1}^{r}\sum_{k=1}^{j-1}(q_{k,-j}p_{k,-t} - q_{jt})p_{j,-s} + \sum_{j=1}^{r}\sum_{k=1}^{j}(q_{k,-j}p_{k,-s} - q_{js})p_{j,-t} + X_{-s,t} \tag{15}$$

$$\tau_r'(X_{r,r+1}) = q_{r,r+1}$$

$$\tau_r'(X_{r+1,r}) = \sum_{s=r+1}^{n}(\tau_r^*(X_{r+1,s})p_{rs} + p_{r,-s}X_{r+1,-s}) - \sum_{j=1}^{r}(\tau_r'(X_{jr}) + 1)p_{j,r+1} + 2q_{r,-(r+1)}p_{r,-r}$$

where

$$\tau_r^*(X_{r+1,s}) = \sum_{k=1}^{r}q_{k,-(r+1)}p_{k,-s} + X_{r+1,s} + \tfrac{1}{2}r\delta_{r+1,s}$$

and in the case $r = n$

$$\tau_n'(X_{i,-j}) = q_{i,-j}$$

$$\tau_n'(X_{-n,n}) = -\sum_{k=1}^{n-1}\left(2X_{kn} + q_{k,-k} + \sum_{j=k+1}^{n-1}2q_{k,-j}p_{j,-n}p_{k,-n}\right)$$
$$- \sum_{k=1}^{n}4q_{k,-n}p_{k,-n}p_{n,-n} - 4p_{n,-n}X_{nn} + 2(n+1)p_{n,-n}.$$

For any $r = 1, 2, \ldots, n$, the elements b_r have the same meaning as the element b from § 4 of Burdík (1985). Therefore, we can apply theorem 3 of that paper to the realisations (15), thus obtaining the following proposition.

Proposition. τ'_r are Schur realisations of sp(n, R) in the $W_{2N_r} \otimes U(\mathrm{gl}(r, R) \oplus \mathrm{sp}(n-r, R))$ for any $r = 1, 2, \ldots, n$.

4. Discussion

In this section we shall compare our results to the realisations given by Deenen and Quesne (1984b) (see formulae (6.21)). These formulae give for $d = 3$ the realisations of the algebra sp(3, R) which we have rewritten explicitly

$$D^+_{ij} = w_{ij} \qquad i, j = 1, 2, 3$$

$$E_{ij} = \sum_{k=1}^{3} w_{ki}\frac{\partial}{\partial w_{kj}} + w_{ij}\frac{\partial}{\partial w_{jj}} + C_{ij} + \tfrac{1}{2}n\delta_{ij}$$

$$\begin{aligned} D_{33} = {} & 2C_{13}\frac{\partial}{\partial w_{13}} + 2C_{23}\frac{\partial}{\partial w_{23}} + 4\left(C_{33} + \frac{n}{2}\right)\frac{\partial}{\partial w_{33}} + w_{11}\frac{\partial}{\partial w_{13}}\frac{\partial}{\partial w_{13}} + w_{22}\frac{\partial}{\partial w_{23}}\frac{\partial}{\partial w_{23}} \\ & + 2w_{12}\frac{\partial}{\partial w_{13}}\frac{\partial}{\partial w_{23}} + 4w_{13}\frac{\partial}{\partial w_{13}}\frac{\partial}{\partial w_{33}} \\ & + 4w_{23}\frac{\partial}{\partial w_{23}}\frac{\partial}{\partial w_{33}} + 4w_{33}\frac{\partial}{\partial w_{33}}\frac{\partial}{\partial w_{33}}. \end{aligned} \tag{16}$$

If we put $r = n = 3$ in the formulae (14) we obtain

$$\rho_3(X_{i,-j}) = \bar{a}_{i,-j} \qquad i, j = 1, 2, 3$$

$$\rho_3(X_{ij}) = \sum_{k=1}^{3} \bar{a}_{k,-i}a_{k,-j} + \bar{a}_{i,-j}a_{j,-j} + \tilde{X}_{ij}$$

$$\begin{aligned} \rho_3(X_{-3,3}) = {} & -2\tilde{X}_{13}a_{1,-3} - 2\tilde{X}_{23}a_{2,-3} - 4\tilde{X}_{33}a_{3,-3} - \bar{a}_{1,-1}a_{1,-3} \\ & - \bar{a}_{2,-2}a_{2,-3}a_{2,-3} - 2\bar{a}_{1,-2}a_{1,-3}a_{2,-3} - 4\bar{a}_{1,-3}a_{1,-3}a_{3,-3} \\ & - 4\bar{a}_{2,-3}a_{2,-3}a_{3,-3} - 4\bar{a}_{3,-3}a_{3,-3}a_{3,-3}. \end{aligned} \tag{17}$$

The sp(3, R)-algebra automorphism φ: $\varphi(X_{ij}) = E_{ij}$, $\varphi(X_{i,-j}) = D^+_{ij}\varphi(X_{-i,j}) = -D_{ij}$, and the mapping ψ: $\psi(\bar{a}_{i,-j}) = w_{ij}$, $\psi(a_{-i,j}) = \partial/\partial w_{ij}$, $\psi(\tilde{X}_{ij}) = C_{ij} + \frac{1}{2}n\delta_{ij}$, evidently transform the formulae (17) to the formulae (16).

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